Linear Algebra
Extra Material: Rows and Columns

G-02

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As we saw in the lecture, there are two ways to think about linear combinations of vectors: the column picture and the row picture. In the following we are going to prove a general result about these mental pictures, to which we will actually come back a few weeks later in an even more general setting.

Here we are particularly interested in what happens when the rows of the linear combination are just a scalar multiple of each other in the row picture. Take as an example the two vectors

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

For $\lambda, \mu \in \mathbb{R}$, the linear combinations of these vectors can be written as

$$\lambda \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot u_2 \\ u_2 \end{pmatrix}$$

which can be further rewritten in the extended form:

$$\begin{aligned} 4\lambda - 4\mu &= 2 \cdot u_2 \\ 2\lambda - 2\mu &= u_2 \end{aligned}$$

We can see that the rows are just a scalar multiple of each other. If you multiply $(2\lambda - 2\mu = u_2)$ with 2 you get the first row.

It is also true that the vectors above are collinear. Now the question is: is this always the case? In other words: ***If the rows in the row picture are a scalar multiple of each other, does that always mean that the vectors in the column picture are also a scalar multiple of each other?***

The answer is yes and we can prove it!

Proof. (in 2 dimensions)
We want to have rows that are a scalar multiple of each other. So let $x, y, u \in \mathbb{R}$ arbitrary scalars and $\lambda, \mu, \in \mathbb{R}$ such that

$$\lambda \cdot x + \mu \cdot y = u$$

This is going to be our first row. Now let $k \in \mathbb{R}$ be a real number. Since we know that the second row is a scalar multiple of the first row we can write it as

$$k \cdot (\lambda \cdot x) + k \cdot (\mu \cdot y) = k \cdot u$$

See how this is just a scalar multiple of the first equation. Now we can read the vectors from these equations:

$$\begin{aligned} \lambda \cdot x + \mu \cdot y &= u \\ k \cdot (\lambda \cdot x) + k \cdot (\mu \cdot y) &= k \cdot u \end{aligned}$$

So our vectors are $\mathbf{v} = \begin{pmatrix} x \\ k \cdot x \end{pmatrix}, \mathbf{t} = \begin{pmatrix} y \\ k \cdot y \end{pmatrix}$. We want to show that $\mathbf{v}$ and $\mathbf{t}$ are a scalar multiple of each other. This means we want to find a scalar $\alpha \in \mathbb{R}$ that satisfies $\mathbf{v} = \alpha \cdot \mathbf{t}$ for arbitrary $\mathbf{v}$ and $\mathbf{t}$. But this wouldn't work if $\mathbf{t} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Therefore we make a *case distinction*:

1. *Case 1:* $y \neq 0$ and hence $\mathbf{t} \neq \mathbf{0}$

   We are now looking for α such that $\mathbf{v} = \alpha \cdot \mathbf{t}$, this means:

   $$\begin{pmatrix} x \\ k \cdot x \end{pmatrix} = \alpha = \begin{pmatrix} y \\ k \cdot y \end{pmatrix}$$

   This gives us 2 equations:

   $$\begin{aligned} x &= \alpha \cdot y \\ k \cdot x &= \alpha \cdot (k \cdot y) \end{aligned}$$

Reformulating the first equation by dividing it by y gives us: $\alpha = \frac{x}{y}$. *We can divide by y because we assumed in this case that it can not be equal to 0.* This is an important statement and you should specify that the variable that you divide by can not be equal to 0.

Now you can also check that $\alpha = \frac{x}{y}$ also satisfies the second equation by substituting the value of α :

$$k \cdot x = \frac{x}{y} \cdot (k \cdot y) = x \cdot k \cdot \frac{y}{y} = x \cdot k$$

This holds and hence we found an α that satisfies $\mathbf{v} = \alpha \cdot \mathbf{t}$ for the case when $y \neq 0$. This means the vectors $\mathbf{v}$ and $\mathbf{t}$ are collinear in this case. Let's look at the other case.

2. *Case 2:* $y = 0$ and hence $\mathbf{t} = \mathbf{0}$

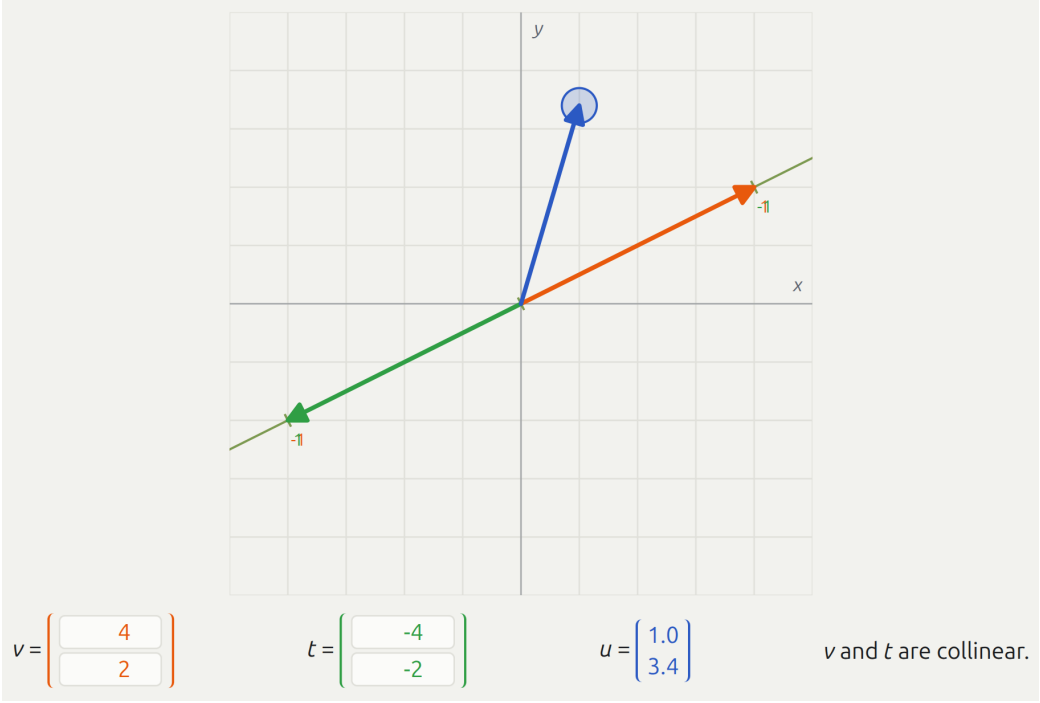
   Now we can not find α such that $\mathbf{v} = \alpha \cdot \mathbf{t} = \mathbf{0}$ unless $\mathbf{v} = \mathbf{0}$. But we can find α such that $\alpha \cdot \mathbf{v} = \mathbf{t} = \mathbf{0}$. For any $\mathbf{v}$ multiplying it by $\alpha = 0$ gives us the $\mathbf{0}$ vector. We can also use the fact that $\mathbf{0}$ vector is a scalar multiple of all vectors.

   This shows that $\mathbf{v}$ and $\mathbf{t}$ are a scalar multiple of each other, also when $y = 0$.

Now since for all possible cases $y = 0$ and $y \neq 0$ (y is either equal to 0 or not there is no other possibility) we showed that $\mathbf{v}$ and $\mathbf{t}$ are a scalar multiple of each other, we are done with the proof. In all cases, if the rows in the row picture are a multiple of each other, then the vectors are a multiple of each other in the column picture.

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You can see this visually in the following example. If the rows are a scalar multiple of each other then the columns are also a scalar multiple of each other. (And we can only represent vectors $\mathbf{u}$ that are on the single line that is spanned by $\mathbf{v}$ and $\mathbf{t}$.)



This is actually just one case of a more general result. We have proven this for 2 dimensions but this is true for all dimensions. We are going to see in the following weeks how we can prove the more general case and how thinking in terms of matrices is going to help us with this problem.

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